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Scattering of Pions by Light Nuclei*

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The scattering of pions by light nuclei is calculated using an approximate, high-energy, small-angle multiple-scattering expansion which neglects off-the-energy-shell scattering. The approximations needed to obtain this expansion from an exact multiple-scattering theory are examined. It is found that the unknown contribution of the off-the-energy-shell scattering makes any calculation of pion-nucleus scattering unreliable for large angles. Using pion-nucleon phase shifts and electron-scattering data, results are obtained for the scattering of pions of about 80 MeV by lithium, carbon, and oxygen.

I. INTRODUCTION

SOME calculations[1–3] of the scattering of pi mesons by nuclei have used the impulse or single-scattering approximation, without corrections for Coulomb or multiple scattering. Others[4–7] have been based on the optical model. The optical potential V_c is obtained by approximately summing a formal multiple-scattering expansion, and the resulting differential equation is solved numerically.

In this paper we will evaluate corrections to the impulse term with a high-energy, small-angle approximation obtained from a complete multiple-scattering expansion. The method, which is basically that of Glauber,[8] gives analytic results in reasonable agreement with the measured scattering[2,4,5] of pions of about 80 MeV by lithium, carbon, and oxygen; the parameters used are those derived from pion-nucleon and electron-scattering data.

The optical model has the advantage of not making these approximations. A minor disadvantage of the optical model is the necessity of solving numerically; however, modern computers are quite adequate for this task.

Nevertheless, we will see that the high-energy and small-angle approximations are reasonable in this problem for angles up to the diffraction minimum at $\theta_0 \sim 70°$. A more serious limitation is the omission of off-the-energy-shell scattering. Little is known about the behavior of the two-body scattering amplitude $(\mathbf{q}'|t|\mathbf{q})$ for $|\mathbf{q}'| \neq |\mathbf{q}|$; the Glauber method neglects it completely. In the optical model, a definite choice for $(\mathbf{q}'|t|\mathbf{q})$ off the energy shell is required in order to calculate $V_c(r)$. The contribution of the off-shell amplitude to the cross section is sensitive to the specific model assumed, and is substantial. Since it tends to drop off more slowly than the on-the-energy-shell scattering, this makes *all* calculations of the large-angle pion-nucleus scattering unreliable. An advantage of our procedure is that the various contributions to the total amplitude are separated and can be studied independently; if one has a model for off-shell scattering, this can be included also.

We begin in Sec. II by summarizing the approximate multiple-scattering theory due to Glauber and a minor modification to the theory. In Sec. III we obtain this from the exact multiple-scattering formalism of Watson[9] and in Sec. IV we present the actual calculations and results. We conclude with an Appendix which discusses some of the approximations in detail.

II. THE HIGH-ENERGY APPROXIMATION

Glauber[8] has developed an approximate method for calculating high-energy, small-angle scattering. He considers a wave $\phi_q = e^{iqz}$ incident upon a potential V of magnitude V_0 and range R. If $V_0/E(q) \ll 1$ and $qR \gg 1$, little reflection or refraction occurs at the boundary, and the wave inside the wall is approximately given by $\psi_q = e^{iq(\mathbf{r})z}$. The corresponding scattering amplitude is proportional to $(\phi, V\psi)$; this gives

$$f(\Delta\mathbf{q}) = \left(\frac{q}{2\pi i}\right) \int d^2b\, e^{-i\Delta\mathbf{q}\cdot\mathbf{b}} [e^{i\chi(\mathbf{b})} - 1], \qquad (2.1)$$

where $\mathbf{b}$ is the impact parameter, $\Delta\mathbf{q} = \mathbf{q}' - \mathbf{q}$, and the integral is over a plane normal to $\mathbf{q}$. The function $\chi(\mathbf{b})$ represents the total phase shift the wave suffers in traversing V. If the potential is known, $\chi(\mathbf{b})$ can be calculated from

$$\chi(\mathbf{b}) = -(1/v) \int_{-\infty}^{\infty} V(\mathbf{b}, z) dz;$$

v is the velocity of the particle. Equation (2.1) is not

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7 R. M. Frank, J. L. Gammel, and K. M. Watson, Phys. Rev. **101**, 891 (1956).
8 R. J. Glauber, *Lectures in Theoretical Physics* (Interscience Publishers, Inc., New York, 1959), Vol. 1, p. 315.
9 K. M. Watson, Phys. Rev. **105**, 1388 (1957); Rev. Mod. Phys. **30**, 565 (1958); also earlier papers given here.

expected to be accurate for large Δq in view of the approximation for ψ_q.

If $\chi(\mathbf{b})$ has azimuthal symmetry, with the definition

$$\Gamma(\mathbf{b})=\exp[i\chi(\mathbf{b})]-1, \tag{2.2}$$

Eq. (2.1) becomes

$$f(\Delta q)=\left(\frac{q}{i}\right)\int_0^\infty J_0\left[2qb\sin\left(\frac{\theta}{2}\right)\right]\Gamma(b)bdb. \tag{2.3}$$

Here $q\sin\theta$ has been replaced by $2q\sin(\theta/2)$, corresponding to a more symmetric treatment of $\mathbf{q}$ and $\mathbf{q}'$, i.e., placing the $\mathbf{b}$ plane in (2.1) perpendicular to $(\mathbf{q}+\mathbf{q}')/2$ rather than to $\mathbf{q}$.

In this approximation, the scattering of a particle by A fixed scatters at $\mathbf{r}_i$ is obtained by replacing $\chi(\mathbf{b})$ by $\sum\chi_i(\mathbf{b}-\mathbf{b}_i)$, which yields

$$\begin{aligned}F(\Delta\mathbf{q},\mathbf{b}_i)&=\left(\frac{q}{2\pi i}\right)\int_0^\infty e^{-i\Delta\mathbf{q}\cdot\mathbf{b}}\{\textstyle\prod_{i=1}^A[1+\Gamma_i(\mathbf{b}-\mathbf{b}_i)]-1\}d^2b\\&=\left(\frac{q}{2\pi i}\right)\int e^{-i\Delta\mathbf{q}\cdot\mathbf{b}}[\textstyle\sum_i\Gamma_i(\mathbf{b}-\mathbf{b}_i)\\&\quad+\textstyle\sum_{i<j}\Gamma_i(\mathbf{b}-\mathbf{b}_i)\Gamma_j(\mathbf{b}-\mathbf{b}_j)\\&\quad+\textstyle\sum_{i<j<k}\Gamma_i(\mathbf{b}-\mathbf{b}_i)\Gamma_j(\mathbf{b}-\mathbf{b}_j)\Gamma_k(\mathbf{b}-\mathbf{b}_k)+\cdots]d^2b.\end{aligned}$$

This can be interpreted as a single-scattering term, a double-scattering term, etc.

If the scatterers are nucleons, the expectation value of $F(\Delta\mathbf{q},\mathbf{b}_i)$ for the nuclear ground state yields the elastic-scattering amplitude. Neglecting correlations,[10] i.e., setting

$$\langle\Gamma_i(\mathbf{b}-\mathbf{b}_i)\Gamma_j(\mathbf{b}-\mathbf{b}_j)\rangle=\langle\Gamma_i(\mathbf{b}-\mathbf{b}_i)\rangle\langle\Gamma_j(\mathbf{b}-\mathbf{b}_j)\rangle,$$

Glauber obtains finally

$$F(\Delta q)=\left(\frac{q}{i}\right)\int_0^\infty J_0\left[2qb\sin\left(\frac{\theta}{2}\right)\right]S(b)bdb, \tag{2.4}$$

where

$$S(b)=\textstyle\sum_i\langle\Gamma_i\rangle+\sum_{i<j}\langle\Gamma_i\rangle\langle\Gamma_j\rangle+\sum_{i<j<k}\langle\Gamma_i\rangle\langle\Gamma_j\rangle\langle\Gamma_k\rangle+\cdots. \tag{2.4a}$$

We will see in the next section that a better approximation is the "modified" Glauber expansion,

$$S(b)=\textstyle\sum_i\langle\Gamma_i\rangle+2^{-1}\sum_i\sum_{j\neq i}\langle\Gamma_i\rangle\langle\Gamma_j\rangle+2^{-2}\sum_i\sum_{j\neq i}\sum_{k\neq j}\langle\Gamma_i\rangle\langle\Gamma_j\rangle\langle\Gamma_k\rangle+\cdots. \tag{2.4b}$$

Equations (2.4) will form the basis of our calculation. The two forms for $S(b)$ differ in the third and succeeding terms by considerable factors; if all the $\langle\Gamma_i\rangle$ are equal, the ratio of the triple scattering term of Eq. (2.4b) to that of (2.4a) is $(\frac{3}{2})\,(A-1)/(A-2)$.

The function $\langle\Gamma_i(b)\rangle$ is needed to evaluate $F(\Delta q)$. If the nucleon density is $\rho(\mathbf{r})$,

$$\langle\Gamma_i(\mathbf{b})\rangle=\int\Gamma_i(\mathbf{b}-\mathbf{b}_i)\rho(\mathbf{b}_i)d^2b_i, \tag{2.5a}$$

where

$$\rho(\mathbf{b}_i)=\int_{-\infty}^{\infty}\rho(b_i,z_i)dz_i,\quad\int\rho(\mathbf{r})d^3r=1.$$

This can be rewritten as

$$\langle\Gamma_i(b)\rangle=(2\pi)^{-2}\int e^{i\Delta\mathbf{q}\cdot\mathbf{b}}\Gamma_i(\Delta\mathbf{q})\rho(\Delta\mathbf{q})d^2\Delta q, \tag{2.5b}$$

where

$$\begin{aligned}\Gamma(\Delta\mathbf{q})&=\int e^{-i\Delta\mathbf{q}\cdot\mathbf{b}'}\Gamma(\mathbf{b}')d^2b'=(2\pi i/q)f(\Delta\mathbf{q}),\\\rho(\Delta\mathbf{q})&=\int{}^{-i\Delta\mathbf{q}\cdot\mathbf{b}'}\rho(\mathbf{b}')d^2b'.\end{aligned} \tag{2.6}$$

Glauber's conditions of $qR\gg1$ and $V_0/E(q)\ll1$ are not well satisfied in our problem. However, Eqs. (2.4) give a connection between two-body scattering amplitude f and the many-body amplitude F which may be accurate even if f itself cannot be adequately obtained by these methods. Glauber has pointed out that the single scattering term in F is identical to that obtained from the impulse approximation, which is valid under less restrictive assumptions, e.g., for large angles. We will now examine the validity of the higher order scattering terms by using Watson's exact multiple-scattering formalism.

III. EXACT MULTIPLE-SCATTERING THEORY

In a series of papers, Watson[9] has developed an exact multiple-scattering theory. We will briefly summarize its essentials, and then make the approximations needed to obtain Eqs. (2.4) from this theory.

Consider a particle incident upon a nucleus of A nucleons. The Hamiltonian is

$$H=[H_N+h]+\sum_{i=1}^{A}V_i\equiv H_0+V, \tag{3.1}$$

where H_N is the nuclear Hamiltonian, h is the incident particle's kinetic energy operator, and V_i is the two-body interaction. The scattering of the particle is given by

$$T=V+Va^{-1}T, \tag{3.2}$$

where

$$a=E_a-H_0+i\epsilon,$$

and $E_a=E(q)+W_\gamma$, where W_γ is the initial energy of the nucleus.

[10] For a discussion of this approximation, see Ref. 8, p. 394; also M. A. B. Beg, Phys. Rev. **120**, 1867 (1960); R. J. Glauber, Physica **22**, 1185 (1956).

A formal solution of Eq. (3.2) is

$$T=\sum_i t_i'+\sum_i\sum_{j\neq 1} t_i'a^{-1}t_j' + \sum_i\sum_{j\neq i}\sum_{k\neq j} t_i'a^{-1}t_j'a^{-1}t_k'+\cdots, \quad (3.3)$$

where the bound two-body amplitude t_i' satisfies

$$t_i'=V_i+V_ia^{-1}t_i'. \quad (3.4)$$

The terms of Eq. (3.3) correspond to sequences of scatterings such that no two successive scatterings are due to the same nucleon.

The elastic scattering amplitude T_c is the "coherent part" of T. For an operator O, the coherent part O_c is defined by

$$\begin{aligned}\langle \mathbf{q}'\gamma'|O_c|\mathbf{q}\gamma\rangle &= \langle \mathbf{q}'\gamma|\,|\mathbf{q}\gamma\rangle \quad \text{if} \quad W_\gamma=W_{\gamma'},\\ &=0 \quad \text{if} \quad W_\gamma\neq W_{\gamma'}.\end{aligned}$$

Hence

$$T_c=\sum t_{ic}'+\sum_i\sum_{j\neq i}(t_i'a^{-1}t_j')_c+\cdots. \quad (3.5)$$

The optical potential is defined by

$$T_c=V_c+V_ca^{-1}T_c. \quad (3.6)$$

Neglecting excited nuclear states gives

$$(t_i'a^{-1}t_j'a^{-1}t_k'\cdots)_c\approx t_{ic}'a^{-1}t_{jc}'a^{-1}t_{kc}'\cdots; \quad (3.7a)$$

for large A,

$$\sum_i\sum_{j\neq i}\cdots t_i'a^{-1}t_j'\cdots\approx\sum_i\sum_j\cdots t_i'a^{-1}t_j'\cdots. \quad (3.7b)$$

Equation (3.5) now implies

$$T_c=\sum t_{ic}'+\sum t_{ic}'a^{-1}T_c.$$

Hence, in this approximation, Watson obtains

$$\langle \mathbf{q}'|V_c|\mathbf{q}\rangle=\sum_i\langle \mathbf{q}'|t_{ic}'|\mathbf{q}\rangle, \quad (3.8)$$

where

$$\langle \mathbf{q}'|t_{ic}'|\mathbf{q}\rangle\approx\langle \mathbf{q}'|t_i|\mathbf{q}\rangle\rho(\Delta\mathbf{q}). \quad (3.9)$$

The replacement of t' by the free two-body amplitude, t, which involves neglecting the effects of nucleon binding on t, is essentially the impulse approximation.

Two models which have been applied to pion-nucleus scattering are[4,5,7,11]

$$\langle \mathbf{q}'|t|\mathbf{q}\rangle\simeq\langle \mathbf{q}|t|\mathbf{q}\rangle=\gamma, \quad (3.10a)$$

and[5,6]

$$\langle \mathbf{q}'|t|\mathbf{q}\rangle\simeq\alpha+\delta(\mathbf{q}-\mathbf{q}'), \quad (3.10b)$$

where γ, α, and δ are complex constants known from pion-nucleon scattering. These lead to

$$\langle \mathbf{r}|V_c\psi\rangle\sim A\gamma\rho(\mathbf{r})\psi(\mathbf{r}) \quad (3.11)$$

and

$$\langle r|V_c\psi\rangle\sim A\alpha\rho(\mathbf{r})\psi(\mathbf{r})+A\delta\boldsymbol{\nabla}[\rho(\mathbf{r})\nabla\psi(\mathbf{r})], \quad (3.12)$$

respectively. The first model assumes Eq. (3.8) is dominated by $\rho(\Delta q)$, and that the variation in t is relatively unimportant; the second attempts to take into account the dominant p-wave character of the two-body interaction. Equation (3.10b) is a good approximation for t (on the energy shell) up to about 100 MeV. However, the first model cannot be made to fit the data at all angles. The second model, in a somewhat altered form, fits if the six parameters used (real and imaginary parts of α and δ, nuclear radius and surface thickness) are varied sufficiently from the usual values.

These models must assume that (3.10a) or (3.10b) holds if $|\mathbf{q}|\neq|\mathbf{q}'|$. There is no reason to believe that this is true.

To obtain the approximation discussed in Sec. II, we begin with Eq. (3.5) and make the approximation (3.7a). Also, we write

$$\frac{1}{E(q)-E(q'')+i\epsilon}=P\frac{1}{E(q)-E(q'')}-i\pi\delta[E(q)-E(q'')]$$

and neglect the first term, corresponding to off-the-energy-shell scattering. Thus we obtain

$$\begin{aligned}(\mathbf{q}'|T_c|\mathbf{q})&=\sum_i\langle \mathbf{q}'|t_{ic}'|\mathbf{q}\rangle\\ &\quad -i\pi\sum_{i\neq j}\int\langle \mathbf{q}'|t_{ic}'|\mathbf{q}''\rangle\langle \mathbf{q}''|t_{jc}'|\mathbf{q}\rangle\\ &\qquad\times\delta[E(q)-E(q')]d^3q''+\cdots\\ &=\sum_i\langle \mathbf{q}'|t_i|\mathbf{q}\rangle\rho(\mathbf{q}-\mathbf{q}')\\ &\quad -i\pi[E(q)/q]\sum_{i\neq j}\int\langle \mathbf{q}'|t_i|\mathbf{q}''\rangle\rho(\mathbf{q}''-\mathbf{q}')\\ &\qquad\times\langle \mathbf{q}''|t_j|\mathbf{q}\rangle\rho(\mathbf{q}''-\mathbf{q})q^2d\Omega''+\cdots.\end{aligned}$$

The angular integrations $q''^2d\Omega''$, $\cdots$, in the second and higher terms are equivalent to integrating d^2q'', $\cdots$, over the surfaces of spheres of radius q. If these spheres are replaced by planes tangent to them at $\frac{1}{2}|\mathbf{q}+\mathbf{q}'|$, and the relation

$$\begin{aligned}\langle \mathbf{q}'|t|\mathbf{q}\rangle&=-f(\Delta q)/(2\pi)^2E(q)\\ &=-q\Gamma(\Delta q)/(2\pi)^3iE(q)\end{aligned}$$

is used, we obtain Eqs. (2.4) and (2.4b).

In the Appendix, the double scattering term is studied in detail for our model. There it is shown that the on-the-energy-shell scattering given by the Glauber and Watson expressions are equivalent in the limiting case of high momentum and small angle, i.e., $[1+\cos(\theta/2)]\approx 2$. For the actual parameters appearing in our carbon calculation, the two differ by 15% or less for *all* angles. This corresponds to about 10% in the cross section for $\theta\lesssim\theta_0$. It also appears that the easily evaluated higher order Glauber terms represent a fair approximation to the corresponding complicated Watson terms. However, since these higher order terms are relatively important for $\theta\gtrsim\theta_0$, the results for large angles are not expected to be very accurate.

The off-shell scattering contribution to the double-scattering amplitude, which is omitted in the Glauber

[11] E. Auerbach (private communication).

approximation, is found to be comparable to the above errors for small angles. However, for large angles, it is of the same order of magnitude as the total on-the-energy-shell amplitude. Since its explicit value is sensitive to delicate cancellations in the integral, it is not possible to do more than estimate its order of magnitude. This means a basic uncertainty of perhaps 10% in the cross section at small angles, and of order 1 at large angles. This is clearly true both for our calculation and for the corresponding optical-model calculation.[3,11a]

IV. CALCULATIONS AND RESULTS

In order to calculate pion-nucleus scattering with Eqs. (2.4), we must have an explicit form for

$$\langle\Gamma_i(\mathbf{b})\rangle=\int\Gamma_i(\mathbf{b}-\mathbf{b}_i)\rho(\mathbf{b}_i)d^2b_i \tag{2.5a}$$

$$=(2\pi)^{-2}\int e^{i\Delta\mathbf{q}\cdot\mathbf{b}}\Gamma_i(\Delta\mathbf{q})\rho(\Delta\mathbf{q})d^2\Delta\mathbf{q}\,, \tag{2.5b}$$

$$\Gamma_i(\Delta\mathbf{q})\equiv(2\pi i/q)f_i(\Delta\mathbf{q})\,. \tag{2.6}$$

The function $\Gamma|\Delta\mathbf{q}|$ is known experimentally only for $|\Delta\mathbf{q}|\leq 2q$; to calculate its transform $\Gamma(\mathbf{b})$ or to evaluate (2.5b) requires that we know it for all $\Delta\mathbf{q}$.

Since (2.5b) contains the form factor $\rho(\Delta\mathbf{q})$ which drops off rapidly for large Δq, the integral is presumably not very sensitive to $\Gamma(\Delta q)$ for large arguments. This suggests substituting into Eq. (2.7b) the S plus P-wave form for pion nucleon scattering,

$$f(\Delta q)=\alpha+\beta\mathbf{q}\cdot\mathbf{q}'/q^2$$
$$=\left[1-\frac{\beta}{2(\alpha+\beta)}\frac{\Delta q^2}{q^2}\right](\alpha+\beta)\equiv c_0+c_1\Delta q^2\,, \tag{4.1}$$

and assuming its validity for all Δq. This we will refer to as model B, and to the corresponding $\langle\Gamma\rangle$ as $\langle\Gamma_B\rangle$. Alternatively, we can argue that $\Gamma_i(\mathbf{b})$ has a short range; if the nucleus is large enough, its precise form is unimportant. We assume for convenience that Γ has the form

$$\Gamma_A(\mathbf{b})=\gamma\exp[-b^2/l^2]\,,$$

and fit its transform $\Gamma(\Delta q)$ to Eq. (4.1) for small Δq. This leads to

$$\gamma=q(\alpha+\beta)^2/\beta\,,$$
$$l^2=2\beta/q^2(\alpha+\beta)\,,$$

and

$$\Gamma_A(\Delta\mathbf{q})=(\alpha+\beta)\exp[-\Delta q^2/2q^2(\alpha+\beta)]\,. \tag{4.2}$$

[The same result could have been obtained by "exponentiating" Eq. (4.1).] This will be referred to as model A.

Note that while $\Gamma_A(\Delta\mathbf{q})$ and $\Gamma_B(\Delta\mathbf{q})$ have the same $\Delta\mathbf{q}$ dependence for small $\Delta\mathbf{q}$, Γ_A goes to zero rapidly for $\Delta ql\gg1$ and Γ_B diverges. Nevertheless the corresponding cross sections are essentially the same for $\theta\leq\theta_0$, verifying that the nuclear density does dominate in $\langle\Gamma\rangle$.

The lithium, carbon, and oxygen nuclei are $(1p)$ shell nuclei and are well described by a density function[12]

$$\rho(r)=\rho_0[1+(Z-2)r^2/3a^2]\exp(-r^2/a^2)\,, \tag{4.3}$$

or

$$\rho(\Delta q)=(1+d_1\Delta q^2)\exp(-\tfrac{1}{4}\Delta q^2a^2)\,,$$
$$d_1=-(Z-2)a^2/6Z\,,$$

where a is about 1.6 F according to electron-scattering data. Thus, Eq. (2.7b) gives

$$\langle\Gamma_B(\mathbf{b})\rangle=\left(\frac{i}{2\pi q}\right)\int e^{i\Delta\mathbf{q}\cdot\mathbf{b}}(c_0+c_1\Delta q^2)\times(1+d_1\Delta q^2)e^{-\Delta q^2a^2/4}d^2\Delta q$$
$$=\left(\frac{i}{q}\right)\int J_0(\Delta qb)(c_0+c_1\Delta q^2)\times(1+d_1\Delta q^2)e^{-\Delta q^2a^2/4}\Delta qd\Delta q$$
$$=\sum_{i=1}^{3}s_ib^{2i-2}e^{-b^2/a^2}\,. \tag{4.4}$$

Here we have used

$$I(b,l^2,\mu)\equiv\int_0^\infty J_0(yb)e^{-y^2l^2}y^{2\mu-1}dy$$
$$=\frac{1}{2l^{2\mu}}[(d/dx)^{\mu-1}(x^{\mu-1}e^x)]_{x=-b^2/4l^2} \tag{4.5}$$

and have defined the s_i by

$$s_1=(2i/qa^2)[c_0+4(c_1+c_0d_1)/a^2+32c_1d_1/a^4]\,,$$
$$s_2=-(8i/qa^6[(c_1+c_0d_1)+16c_1d_1/a^2]\,, \tag{4.6}$$
$$s_3=(32i/qa^{10})[c_1d_1]\,.$$

Since we have to treat both protons and neutrons, there are two $\langle\Gamma_B\rangle$'s, with s_i^p and s_i^n, respectively.

Substituting Eq. (4.4) into Eqs. (2.4) gives after some algebra

$$F=\sum F_n\,,\quad F_n=\sum_{l=0}^{n}F_{l,n-l}\,, \tag{4.7}$$

$$F_{l,m}=(q/i)\sum_{i=1}^{3n+1}Q(l,m,i)I(\Delta q,\,l/a^2+m/a^2,\,i)\,. \tag{4.8}$$

Here F_n is the amplitude for n-fold scattering; $F_{l,m}$ is the amplitude for scattering by l protons and m neutrons;

[11a] *Note added in proof.* It is possible to obtain Glauber's results by using an approximate form of the momentum space Green's function, i.e., by using

$$\mathbf{q}''=\mathbf{q}+\mathbf{n},\quad|\mathbf{n}|\ll|\mathbf{q}|$$

or

$$E(q)-E(q'')+i\epsilon\approx-(\mathbf{q}\cdot\mathbf{n}/2\mu)+i\epsilon.$$

One finds then that the contribution due to off-the-energy-shell scattering by nucleon 1 and then by nucleon 2 is exactly cancelled by the sequence 2,1. Thus, the off-shell scattering is not neglected but rather vanishes in this linear approximation. I would like to thank Professor Glauber for a useful discussion of this point.

[12] D. G. Ravenhall, Rev. Mod. Phys. 30, 430 (1958).

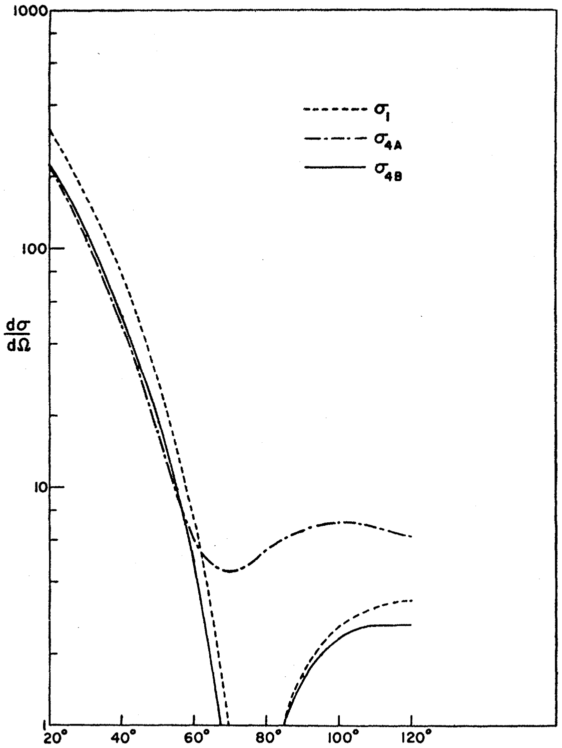

FIG. 1. 80-MeV π^- on carbon, $a=1.6$ F, Anderson phase shifts. The impulse approximation and multiple-scattering corrections through fourfold scattering using models A and B and the Glauber series (2.4a). Cross sections are in millibarns.

I is defined in Eq. (4.5). For the original Glauber series S defined by Eq. (2.4a), the Q's satisfy the recursion relations

$$Q(l,m,i)=[(Z-l+1)/l]\sum Q(l-1,m,i-j+1)s_j{}^p$$
$$=[(N-m+1)/m]\sum Q(l_1,m-1,i-j+1)s_j{}^n,\quad l,m\geq 1.\quad (4.9)$$

Here j is summed to the lesser of i and 3. Also,

$$\begin{aligned} Q(0,0,i)&=0,\\ Q(1,0,i)&=Zs_i{}^p,\quad i\leq 3\\ &=0,\quad i>3 \\ Q(0,1,i)&=Ns_i{}^n,\quad i\leq 3\\ &=0,\quad i>3. \end{aligned}\quad (4.10)$$

If S is defined by the modified Glauber series, Eq. (2.4b), then the factors in brackets in Eqs. (4.9) become complicated. An approximation which gives an error of order F_n/A for $n\geq 3$ is obtained with the replacement of these factors for $n\geq 3$ by $[(Z-1+\delta_{1,l})(l+m)/2l]$ and $[(N-1+\delta_{1,m})(l+m)/2m]$, respectively.

Similar results are obtained with model A with

$$c_0\rightarrow c_0,\quad c_1\rightarrow 0,\quad a^2\rightarrow a^2+l^2.$$

The single scattering or impulse term $\sum F_i(\Delta q)\rho(\Delta q)$ is taken to be the same as in model B; the function Γ_A is used only to evaluate the multiple-scattering corrections.

So far we have not discussed Coulomb effects or kinematics. If the Coulomb potential is included in Sec. II, an additional $\Gamma_i=\Gamma_{\rm Coul}$ appears in the multiple-scattering expansion, Eqs. (2.4). We retain $\Gamma_{\rm Coul}$ only in the single-scattering term. Since the pion mass is not negligible compared to the nucleon mass, we evaluate the impulse term for the pion-nucleon center of mass frame; since the remaining terms are smaller and more isotropic, we treat the entire amplitude in this way.[13]

For π^--p scattering, neglecting the spin flip term,

$$q\alpha_p=(a_3+2a_1)/3,$$
$$q\beta_p=(2a_{33}+a_{31}+4a_{13}+2a_{11})/3;$$

for π^--n,

$$q\alpha_n=a_3,$$
$$q\beta_n=2a_{33}+a_{31}.$$

Here $a_i=\exp(i\delta_i)\sin\delta_i$, and the $\delta_{2T,2J}$ are the usual phase shifts. For π^+ scattering, $\alpha_p\leftrightarrow\alpha_n$ and $\beta_p\leftrightarrow\beta_n$. We have used the δ_i obtained[14] from Anderson's empiri-

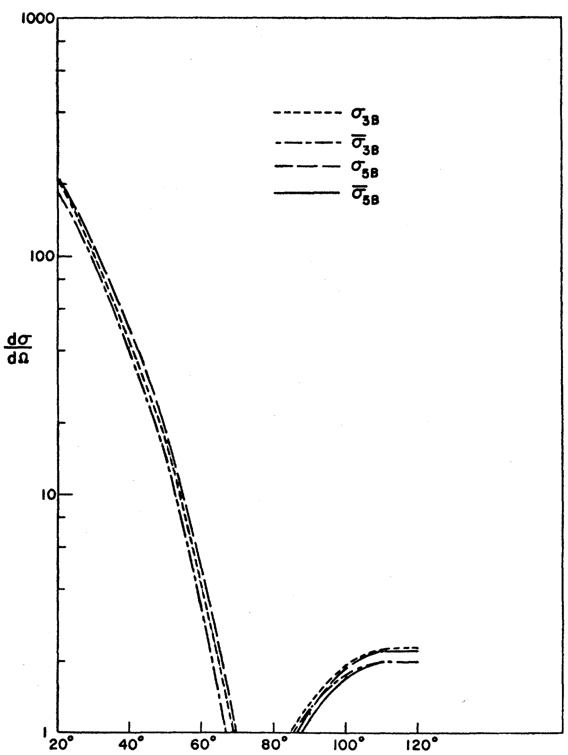

FIG. 2. 80-MeV π^- on carbon, $a=1.6$ F, Hull-Lin phase shifts. Modified Glauber series ($\bar{\sigma}$) and unmodified (σ) for model B. $\bar{\sigma}_{5B}$ and σ_{5B} differ by 1% or less for $\theta<70°$; the former is shown here only for $\theta>85°$.

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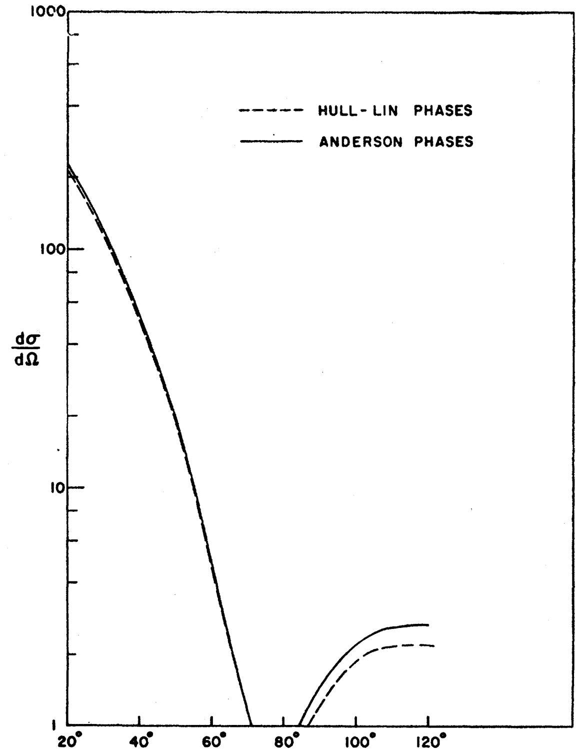

FIG. 3. 80-MeV π^- on carbon, $a=1.6$ F. σ_{5B} plotted with Anderson and Hull-Lin phase shifts.

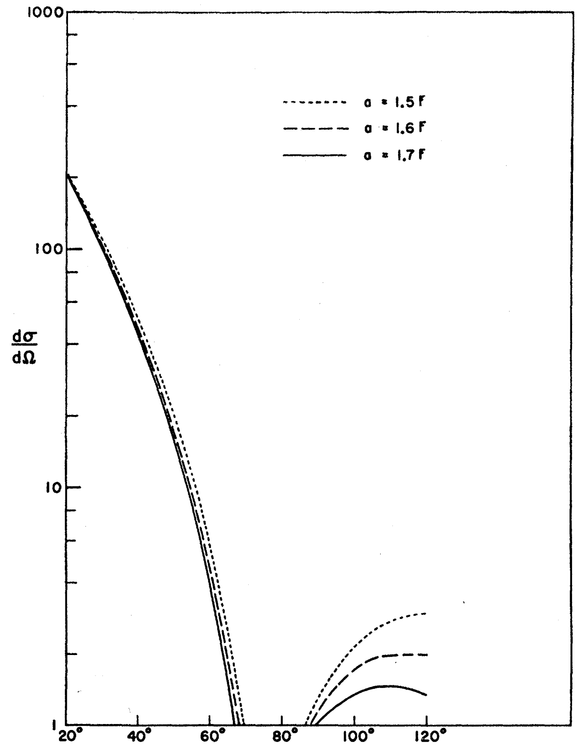

FIG. 4. 80-MeV π^- on carbon, Hull-Lin phase shifts. $\bar{\sigma}_{5B}$ for radius parameter $a=1.5$, 1.6, 1.7 F.

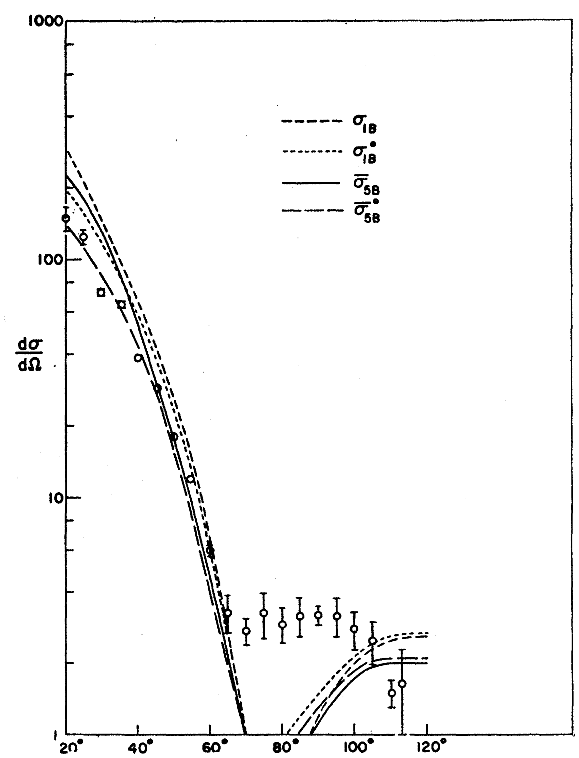

FIG. 5. 80-MeV π^- on carbon, $a=1.6$ F. Hull-Lin phase shifts. Cross sections without (σ^0) and with (σ) the Coulomb amplitude. Note tendency of Coulomb and multiple-scattering effects to cancel. Data are from Baker *et al.* (Ref. 5).

cal formula and also a newer set found by Hull and Lin which differs slightly in the energy range under consideration.

The results are remarkably insensitive to the choice of model A or B, the Glauber or modified Glauber series, and the Anderson or Hull-Lin phases. Let σ_{nA} be the cross section for model A including all terms in the Glauber series through n-fold scattering, let $\sigma_{nA}{}^0$ be the same cross section without the Coulomb amplitude, and similarly for model B. Let $\bar{\sigma}$ be the same quantity for the modified Glauber series. Figures 1 to 4 are plotted for 80 MeV π^- on carbon, with the radius parameter $a=1.6$ F. Figure 1 shows that σ_A and σ_B are nearly equal for $\theta \leq \theta_0$, i.e., both forms for Γ give similar results. Although the different weights in the Glauber and modified Glauber series make σ_{3B} and $\bar{\sigma}_{3B}$ differ, Fig. 2 shows that the total effects nearly cancel. Figure 3 shows that the Hull-Lin phases and Anderson phases give results differing by a few percent.

In Fig. 4 the effect of varying the radius parameter a is seen to be large only for $\theta \gtrsim \theta_0$.

At small angles, Coulomb and multiple-scattering corrections are considerable and tend to cancel for π^- but add for π^+, as is seen in Figs. 5 and 6. Thus a simple impulse approximation fits fairly well for π^- but not π^+, as was found by Williams *et al.*[2]

Figures 7 to 10 make further comparisons of the calcu-

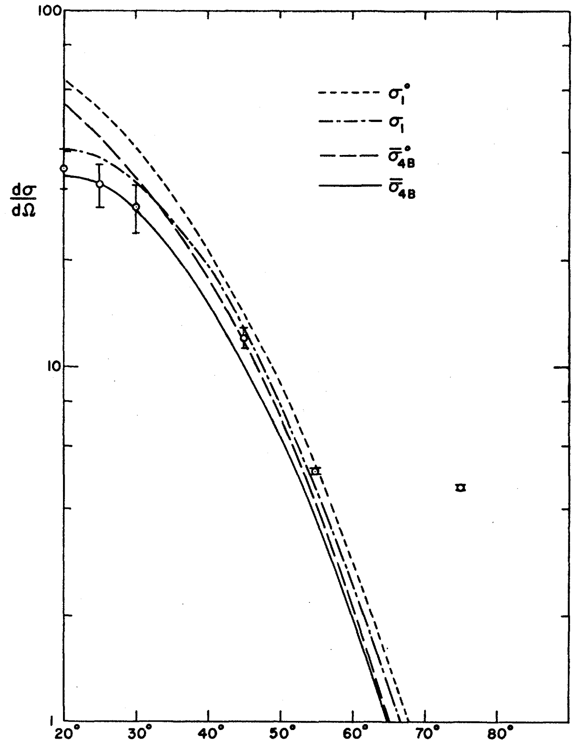

FIG. 6. 78-MeV π^+ on Li, $a=1.7$ F, Hull-Lin phase shifts. Cross sections with and without Coulomb amplitude. Data are from Williams *et al.* (Ref. 2).

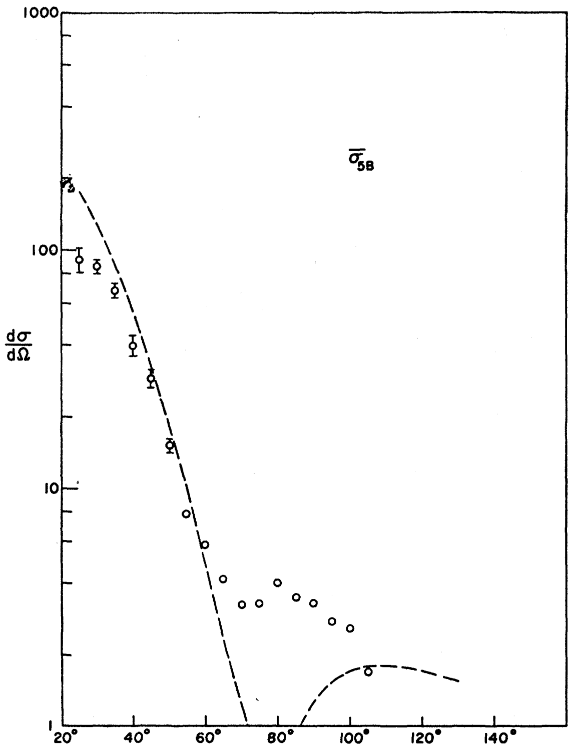

FIG. 8. 87.5-MeV π^- on carbon, $a=1.6$ F, Hull-Lin phase shifts. Data are from Edelstein *et al.* (Ref. 5).

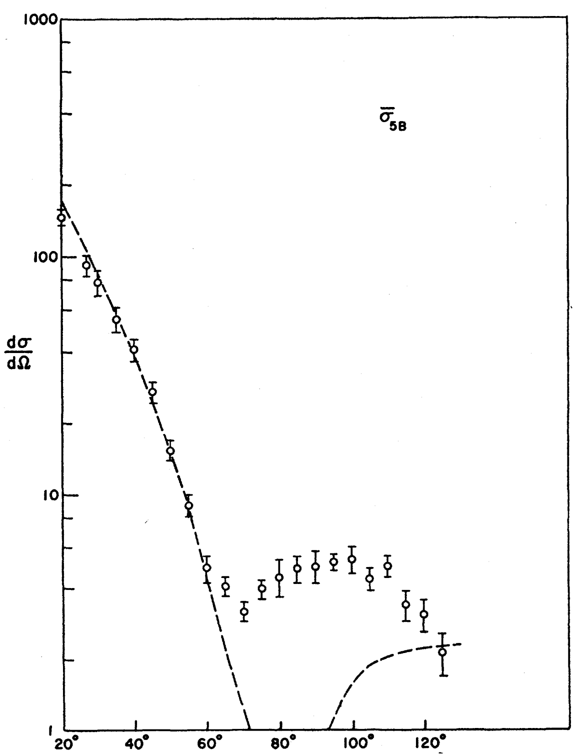

FIG. 7. 69.5-MeV π^- on carbon, $a=1.6$ F, Hull-Lin phase shifts. Data are from Edelstein *et al.* (Ref. 5).

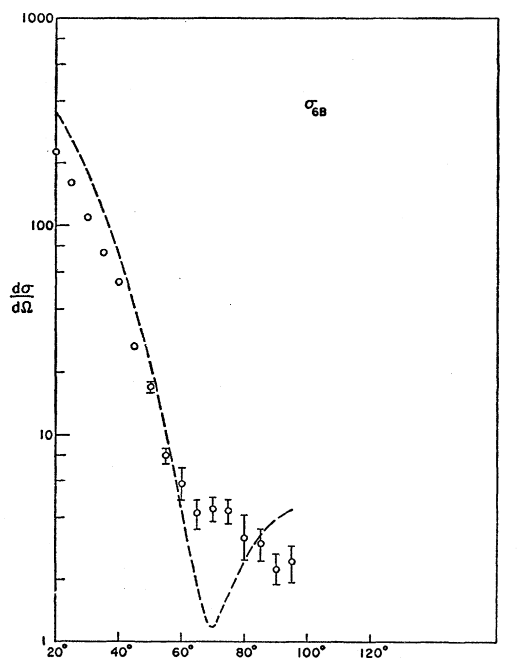

FIG. 9. 87.5-MeV π^- on oxygen, $a=1.6$ F, Anderson phase shifts. σ_{6B} is plotted. It does not differ much from σ_{4B} or σ_{4A}. However, the modified series does not seem to converge well, indicating probably that the multiple-scattering corrections are too crudely approximated for this large a nucleus.

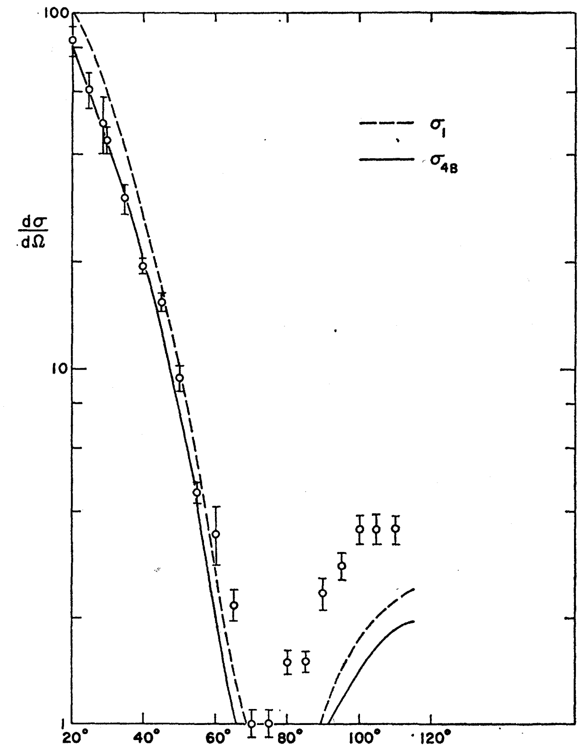

FIG. 10. 78-MeV π^- on lithium, $a=1.6$ F, Hull-Lin phase shifts. Data are from Williams *et al.* (Ref. 3), for $\theta\leq 35°$ and from Baker *et al.* (Ref. 5) for $\theta\geq 40°$.

lated cross sections with experimental data. We note that the calculated curves are in general agreement with the data for $\theta\lesssim\theta_0$, differing by about 20% at some angles for carbon. This is consistent with our estimate of the errors, i.e., about 10% each for the multiple scattering approximations and the unknown off-the-energy-shell scattering amplitude. The fit is better for lithium and poorer for oxygen, as we might expect.

ACKNOWLEDGMENTS

I would like to thank Professor N. M. Kroll for suggesting this problem, and Professor R. E. Peierls, Dr. J. Weneser, and Dr. L. S. Brown for valuable discussions. I am grateful to Professor M. H. Hull and F. Lin for allowing me to have their pion-nucleon phase shifts prior to publication. The numerical calculations were performed with the Brookhaven and Yale computers.

APPENDIX

In this appendix we will discuss in somewhat more detail the approximations made in the double-scattering terms. Our models A and B are both expressible as

$$\langle \mathbf{q}'|t_c|\mathbf{q}\rangle=\textstyle\sum_i c_i(\Delta q/q)^{2i}\exp(-\Delta q^2R^2)\,. \qquad \text{(A1)}$$

Thus a double-scattering term is of the form

$$\langle \mathbf{q}'|t_c a^{-1}t_c|\mathbf{q}\rangle=\textstyle\sum_{ij} c_ic_jK_{ij}/q^{2(i+j)}\,. \qquad \text{(A2)}$$

K_{ij} is given by

$$K_{ij}=\int(\mathbf{q}'-\mathbf{q}'')^2(\mathbf{q}-\mathbf{q}'')^2e^{-(\mathbf{q}'-\mathbf{q}'')^2R^2}e^{-(\mathbf{q}-\mathbf{q}'')^2R^2}\times(E-E''+i\epsilon)^{-1}d^3q''$$

$$=(-)^{i+j}\left(\frac{\partial}{\partial\lambda'}\right)^i\left(\frac{\partial}{\partial\lambda}\right)^jJ(\lambda,\lambda')\bigg|_{\lambda=\lambda'=R^2}\,, \qquad \text{(A3)}$$

where $E=E(q)$, $E''=E(q'')$, and

$$J(\lambda,\lambda')=\int\exp[-(\mathbf{q}'-\mathbf{q}'')^2\lambda'-(\mathbf{q}-\mathbf{q}'')^2\lambda]\times(E-E''+i\epsilon)^{-1}d^3q''$$

$$=2\pi\exp[-q^2(\lambda+\lambda')]\int_0^\infty\left[\sinh\frac{(2q''A)}{q''A}\right]\times\exp[-q''^2(\lambda+\lambda')](E-E''+i\epsilon)^{-1}q''^2dq''\,, \qquad \text{(A4)}$$

$$A=|\mathbf{q}\lambda+q'\lambda'|\,.$$

J is of the form

$$2\int_0^\infty f(y^2)(z^2-y^2+i\epsilon)^{-1}dy$$

$$=\int_0^\infty f(y^2)\left[P\left(\frac{1}{z^2-y^2}\right)-i\pi\delta(z^2-y^2)\right]dy$$

$$=\int_0^\infty[f(y^2)-f(z^2)]\frac{dy}{z^2-y^2}+f(z^2)P\int_0^\infty\frac{1}{z^2-y^2}dy-(i\pi/2z)f(z^2)\,. \qquad \text{(A5)}$$

The first integral is well behaved, and the second vanishes. Using the nonrelativistic approximation for simplicity, or $(E-E'')=(q^2-q''^2)/2\mu$, we obtain

$$K_{ij}=-4\pi\mu\exp(-2z^2)R^{-(2i+2j+1)}\times\left[\int_0^\infty[F_{ij}(y,x,z)-F_{ij}(z,x,z)](y^2-z^2)^{-1}dy+(i\pi/2z)F_{ij}(z,x,z)\right], \qquad \text{(A6)}$$

where

$$x=|q+q'|R\,,\quad y=q''R\,,\quad z=qR\,,$$

and the first few K_{ij} are

$$F_{00}=\sinh(2yx)\exp(-2y^2)y/x\,,$$

$$F_{10}=F_{01}=[-\sinh(2xy)(y^2-z^2+\tfrac{1}{2})+yx\cosh(2xy)]\times\exp(-2y^2)y/x\,, \qquad \text{(A7)}$$

$$F_{11}=[\sinh(2xy)[(y^2+z^2+\tfrac{1}{2})^2-z^2+y^2x^2]-2yx\cosh(2xy)(\tfrac{1}{4}+y^2+z^2+z^2/x^2)]\times\exp(-2y^2)y/x\,.$$

The corresponding result in the Glauber theory is obtained by integrating the δ-function part of (A3) over a plane tangent at $(\mathbf{q}+\mathbf{q}')/2$ to a sphere of radius q. The result is a set of functions G_{ij} similar to the F_{ij}. G_{00} is obtained from F_{00}, for example, by setting

$$\begin{aligned}\sinh(2yx)&=\sinh(2|q+q'|qR^2)\\&=\sinh[4q^2R^2-\Delta q^2R^2/(1+\cos\theta/2)]\\&\approx\tfrac{1}{2}\exp[4q^2R^2-\tfrac{1}{2}\Delta q^2R^2],\end{aligned}$$

and

$$z/x=qR/|q+q'|R\approx\tfrac{1}{2}.$$

This is clearly a good approximation for $qR\gg1$ and $(1+\cos\theta/2)\approx2$; however, if we plot F_{00} and G_{00} for $qR\sim1$, we find that F_{00} and G_{00} differ by 15% or less for *all* angles. F_{01} and G_{01} differ a bit more; the other terms, which are less important, have not been compared explicitly.

Evaluation of triple scattering terms in the Watson expansion is difficult for this model. However, if t_c has a radius parameter R, $t_ca^{-1}t_c$ has $R/\sqrt{2}$. Thus, $t_ca^{-1}(t_ca^{-1}t_c)$ is qualitatively similar to the above integrals but with one radius reduced. Hence the triple scattering and higher Glauber terms are useful only as rough estimates; for $\theta\gtrsim\theta_0$, where they are relatively important, our calculation is not reliable.

The off-shell scattering, which is omitted in our calculation, is given by the integrals in Eq. (A6). We have evaluated the off-shell parts of K_{00}, K_{01}, and K_{11} numerically. If the integrals are split into $y<z$ and $y>z$ parts, we find that the two are comparable in magnitude and opposite in sign. If the c_i in Eq. (A1) are given a $(q''/q)^n$ dependence and included in the integrals, they change greatly. Typically the off-shell amplitudes for model A are 10 or 20% of the corresponding on-the-energy-shell scattering amplitudes for small θ, but are often of the same order or larger for $\theta\gtrsim\theta_0$. The off-shell amplitudes for model B are somewhat larger.

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Nuclear Structure Studies in the Molybdenum Isotopes with (d,p) and (d,t) Reactions*

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Many new energy levels are located in the odd-A molybdenum isotopes and spin and parity values are assigned. In particular, it is found that the ground-state spins of Mo^{99} and Mo^{101} are both $\frac{1}{2}^+$. Occupancy numbers and relative single-quasiparticle energies for the $2d_{5/2}$, $3s_{1/2}$, $1g_{7/2}$, and $2d_{3/2}$ single-quasiparticle states are obtained. The single-quasiparticle energies for Mo^{93}, which are equal to the single-particle energies because Mo^{92} forms a closed shell, differ only little from those in the isotone Zr^{91}. In spite of this, the quasiparticle energies are much lower and the mixing much stronger in the more neutron-rich molybdenum isotopes than in the corresponding zirconium isotopes. A pairing-force calculation revealed that the comparatively small shift in the single-particle levels between zirconium and molybdenum could not account for this completely different behavior of molybdenum and zirconium.

I. INTRODUCTION

DURING the last few years, many nuclear properties have been successfully described by means of the superconductivity model, or pairing theory.[1–3] For instance, the model accounts for the odd-even mass difference, the energy gap in even-even nuclei, and nuclear transition probabilities. With some refinements it is also possible to calculate the energy of the first excited 2^+ and 3^- states as well as their enhanced transition rates with good accuracy.[4,5] The calculations are based upon a knowledge of the unperturbed energy levels of the average shell-model field. Due to meager experimental information, most calculations have, until now, been based upon theoretical estimates of the position of the single-particle levels. Since the result of the calculations depends very sensitively on the single-particle levels, and since the theoretical estimates do not reproduce the finer details of the single-particle levels very well, experimental information on this point is very valuable. In previous papers from this laboratory, single-particle levels have been located in the zir-

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